Pergamon

Int. J. Solids Structures Vol. 35, No. 15, pp. 1743–1750, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00133–9

NUMERICAL ESTIMATES OF THE STRESS INTENSITY FACTORS FOR PERIODIC CRACK SYSTEMS IN THREE-DIMENSIONAL MEDIUM

J. D. SUZDALNITSKY
QAR, Quality Assurance and Reliability, Technion City, Haifa, 32000, Israel

(*Received* 15 *September* 1996; *in revised from* 16 *May* 1997)

Abstract—A three-dimensional medium with a periodic or biperiodic system of circular cracks under normal loading is considered. The displacements are represented in the form of surface integrals and the problem is transformed to a singular integral equations. The stress intensity factors are determined. © 1998 Elsevier Science Ltd.

NOTATION

x,y,z	rectangular coordinates
r,θ	polar coordinates
v	Poisson's ratio
μ	shear modulus
ω,ω_{kmn}	break surfaces
D	dimension
a	radius of a crack
h	distance between layers
l	distance between centres of cracks
ε,δ	parameters connected with a,h,l
u_1,u_2,u_3	displacements
σ_z	normal stress component
τ_{xz},τ_{yz}	shear stress components
σ_0	load
K	stress intensity factor
$\kappa(\theta)$	normalized stress intensity factor
g_{ij}	components of Green's tensor
G,H,R	kernels in integral representations
$\varphi(\rho)$	solutions of integral equations

1. INTRODUCTION

The results of analysis of a three-dimensional (3D) medium with a single circular or elliptical crack are described in detail in (Liebowitz, 1968). There are some publications, referred to by Qin *et al.* (1995), relating to the interaction between two parallel planar cracks in 3D elasticity. The problem of crack interaction in 3D fracture mechanics has an important applied value, but owing to the difficulties of mathematical and numerical evaluation, progress is limited. Research of the stress state in an infinite elastic sheet with a curvilinear crack or a crack system can be realized in the following manner. Solution of the two-dimensional (2D) biharmonic equation for the stress function presented in contour integral form with density function and kernel, that is a fundamental solution of this equation. The integral form obeys boundary conditions, and we obtain a singular integral equation for the unknown density functions. For summing equations numerical analysis can be used. This method, developed by Kurshin and Suzdalnitsky (1975, 1978), is extended here for evaluation of the stress state of a 3D medium, weakened by a periodic or biperiodic system of planar circular cracks.

2. STATEMENT OF THE PROBLEM

The stress state of the body under load is described by the Lame equation. The general solution of this equation can be represented as a surface integral (Banerjee and Butterfield, 1981)

$$\mathbf{u}(M_0) = \int_\omega \int [g_{ij}(M, M_0)]\varphi(M)\,\mathrm{d}\omega \tag{1}$$

where $\mathbf{u}(M_0) = \{u_1, u_2, u_3\}$ is the displacement vector at the point M_0, $\varphi(M) = \{\varphi_1, \varphi_2, \varphi_3\}$ is a vector of unknown functions to be determined from boundary conditions, $[g_{ij}]$ is Green's tensor with components

$$g_{ij} = (x_i-\xi_i)(x_j-\xi_j)/r^3 + \varepsilon\delta_{ij}/r, i,j = 1,2,3$$

$$r^2 = \sum_{i=1}^{3}(x_i-\xi_i)^2, \tag{2}$$

x_1, x_2, x_3—rectangular coordinates (further $x_1 = x, x_2 = y, x_3 = z$), $\varepsilon = 3-4v$, v is Poisson's ratio and $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$.

Let the 3D infinite medium be weakened by a system of circular coplanar cracks and loaded by a tension effort along the normal to the break surface

$$\sigma_z^\infty = -\sigma_0, \quad \tau_{xz}^\infty = \tau_{yz}^\infty = 0. \tag{3}$$

The break surface itself is not loaded

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0. \tag{4}$$

The origin of rectangular coordinates x, y is located at the centre of one of the cracks, plane x, y coincides with its plane. In these coordinates the break surface ω is defined as

$$\omega = \cup\,\omega_{kmn}, \quad \omega_{kmn} = \{(x-kl_1)^2+(y-ml_2)^2 \leqslant d^2, z = nh\}.$$

Further we shall consider the following cases:

(1) $k = 0, \pm 1, \pm 2, \ldots, m = n = 0$. A periodic crack system with centres on the x-axis, $l_1 = l$ is the period, a is the crack radius (Fig. 1a).

(2) $k, m = 0, \pm 1, \pm 2, \ldots, n = 0, l_1 = l_2 = l$. A biperiodic crack system forming a square lattice (Fig. 1b).

(3) $k,\ m = 0, \pm 1, \pm 2, \ldots, n = 0, \pm 1, \pm 2, \ldots, N,\ l_1 = l_2 = l$. A laminated biperiodic crack system, h is the distance between layers, $2N+1$ is the number of layers (Fig. 1c).

(4) $k = m = 0$, $n = 0, \pm 1, \pm 2, \ldots$. A periodic coplanar crack system with centres on the z-axis (Fig. 1d).

3. THE EQUATION FOR THE BIPERIODIC CRACK SYSTEM

We consider case II in detail. Solution (1) with kernel (2) must obey the boundary conditions (3), (4) according to the relations of the theory of elasticity between stresses and displacements

$$\sigma_{ii} = \lambda\,\mathrm{div}\,\mathbf{u} + 2\mu\partial u_i/\partial x_i, \quad \tau_{ij} = \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i),$$

$$2\mu = E/(1+v)$$

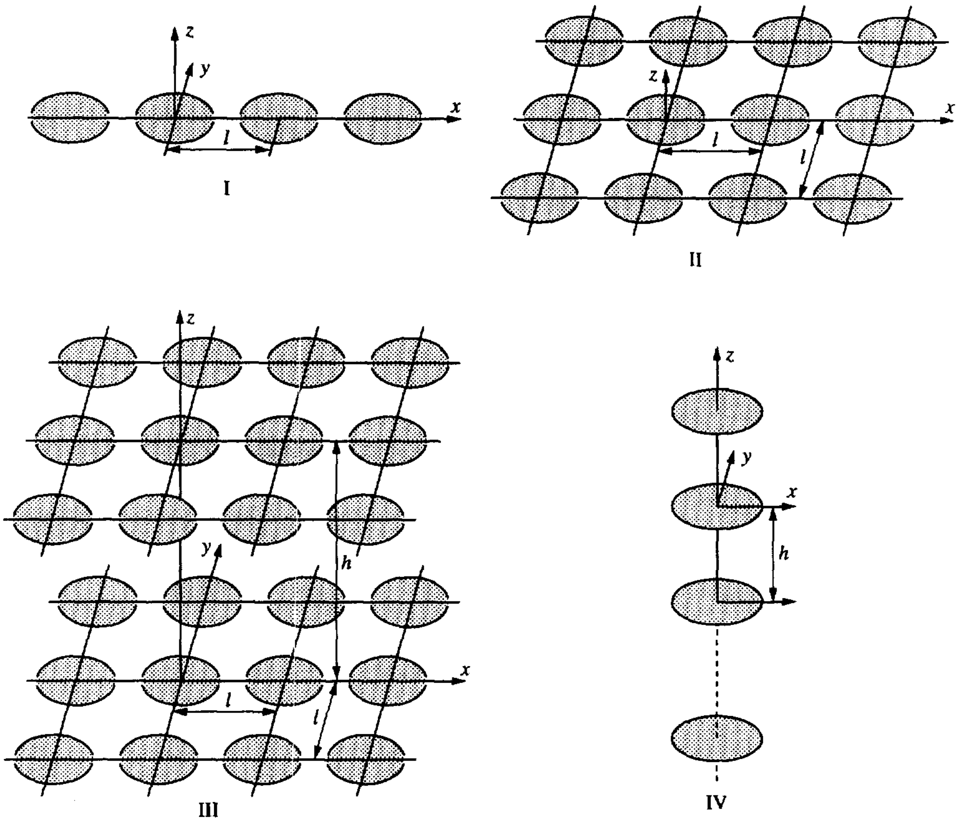

Fig. 1. Crack systems: (I) periodic; (II) biperiodic; (III) laminated biperiodic; and (IV) periodic coplanar.

In this way we obtain three two-dimensional singular integral equations. After conversion to polar coordinates $x = ar\cos\theta, y = ar\sin\theta, \xi = a\rho\cos\theta, \eta = a\rho\sin\tau$ and assumption of $\varphi_1 = \varphi(\rho,\tau)\cos\tau$, $\varphi_2 = \varphi(\rho,\tau)\sin\tau$, $\varphi_3 = 0$ two equations will be realized identically. The third equation is

$$\int_0^1 \rho\,\mathrm{d}\rho \int_\theta^{2\pi+\theta} \sum_{k,m=-\infty}^{\infty} \frac{\partial}{\partial\rho}\frac{1}{R_{km}}\varphi(\rho,\tau)\,\mathrm{d}\tau = -\sigma \tag{5}$$

where

$$R_{km}^2 = R_0^2 - 2(ka_1 + mq_2)l + (k^2+m^2)l^2, \quad R_0^2 = r^2 + \rho^2 - 2r\rho\cos(\theta-\tau),$$

$$q_1 = r\cos\theta - \rho\cos\tau, \quad q_2 = r\sin\theta - \rho\sin\tau, \quad \sigma = \sigma_0(1+\nu)/E(1-2\nu).$$

In the particular case with retention of one member of kernel (4) for $k = m = 0$, eqn (5) has the solution

$$\varphi(\rho,\tau) = \varphi_0(\rho) = \frac{\sigma}{\pi^2}\frac{\rho}{\sqrt{1-\rho^2}}. \tag{6}$$

From eqn (6), it is easy to obtain well-known results for a single circular crack (Liebowitz, 1968).

Let us expand the kernel in eqn (5) in a series with respect to ε, where $\varepsilon = a/l$ is a small parameter

$$\sum_{k,m} \frac{\partial}{\partial \rho} \frac{1}{R_{km}} = \frac{\partial}{\partial \rho} \frac{1}{R_0} + \sum_{j=1}^{\infty} \varepsilon^{2j+1} G_j(r, \theta, \rho, \tau). \tag{7}$$

In particular,

$$G_1 = (3\alpha_{210} + \alpha_{100})(\rho - r\cos(\theta - \tau), \quad \alpha_{nij} = \sum_{k,m} k^{2i} m^{2j} (k^2 + m^2)^{-(n+0.5)}.$$

The solution of eqn (5) we take into consideration in the form

$$\varphi(\rho, \tau) = \sum_{j=0}^{\infty} \varepsilon^j \varphi_j(\rho, \tau). \tag{8}$$

After substituting eqns (7) and (8) in eqn (5) and equating coefficients with equal powers of ε, we obtain a set of integral equations

$$\int_0^1 \rho \, d\rho \int_0^{2\pi} \frac{\partial}{\partial \rho} \frac{1}{R_0} \varphi_0(\rho, \tau) \, d\tau = -\sigma, \tag{9}$$

$$\int_0^1 \rho \, d\rho \int_\theta^{2\pi+\theta} \frac{\partial}{\partial \rho} \frac{1}{R_0} \varphi_j(\rho, \tau) \, d\tau = -\int_0^1 \rho \, d\rho \int_\theta^{2\pi+\theta} F_j(r, \theta, \rho, \tau) \, d\tau, \quad j = 1, 2, \ldots, \tag{10}$$

where

$$F_1 = F_2 = 0, \quad F_{2j+1} = \sum_{k=1}^{j} G_k \varphi_{2(j-k)}, \quad F_{2j+2} = \sum_{k=1}^{j} G_k \varphi_{2(j-k)+1}.$$

In the computations, terms in series (7), (8) containing ε in powers $j \leqslant 7$. It is easily established, that $\varphi_1 = \varphi_2 = \varphi_4 = 0$, $\varphi_j(\rho, \tau) = \varphi_j(\rho)$ for $j = 0$, 3, 5, 6 and $\varphi_7(\rho, \tau) = \varphi_{70}(\rho) + \varphi_{72}(\rho) \cos 4\tau$. A method for solving equation

$$\int_0^a \rho \varphi(\rho) \, d\rho \int_0^{2\pi} \frac{\partial}{\partial \rho} \frac{1}{R_0} \cos 2n\tau \, d\tau = -f(r) \tag{11}$$

for $n = 0$ is well known. Let $f(r) = r^{2k}$, $k = 0, 1, 2, \ldots$ (this kind of $f(r)$ only is of interest here). Solution (11) has the form (Aleksandrov, 1967)

$$\varphi_k(\rho) = \frac{1}{\pi^2} \frac{\rho}{\sqrt{1-\rho^2}} \frac{1}{2k+1} \left\{ (1+\delta_k) a^{2k} - 2k\delta_k \sum_{i=0}^{k-1} C_{k+1}^{i} \frac{\rho^{2i}(a^2-\rho^2)^{k-2i}}{2k-2i-1} \right\}$$

$$\delta_k = (2k)!!/(2k-1)!!, \quad \delta_0 = 0, \quad C_k^i = k!/i!(k-i)!. \tag{12}$$

Function (6) is a particular case of (12) with $k = 2$.

In determining function $\varphi_7(\rho, \tau)$, it is needed to solve the integral equation (11) for $n = 2$ (for a periodic crack system in case I this procedure should be executed for $n = 1$ and $n = 3$ also). For the kernel equation (11) we resort to expansion in an infinite series (Abramovitz and Stegun, 1964).

$$(r^2 + \rho^2 - 2r\rho\cos\tau)^{-1/2} = \frac{1}{r} \sum_{k=0}^{\infty} P_k(\cos\tau) \left(\frac{\rho}{r}\right)^{2k}, \quad \rho < r$$

where $P_k(\cos\tau)$ are Lagrange polynomials.

Equation (11) reduces to the form

$$\int_0^a H_n(r,\rho)\varphi(\rho)\,\mathrm{d}\rho = f(r) \tag{13}$$

where

$$H_n(r,\rho) = \frac{v}{r^2-\rho^2}\sum_{m=0}^{\infty} c_{mn}^{\pm}\left(\frac{\rho}{r}\right)^{\pm 2m},$$

$$c_{0n}^{\pm} = b_{0n}^{\pm},\quad c_{mn}^{\pm} = b_{mn}^{\pm} - b_{m-1,n}^{\pm},$$

$$b_{mn}^{+} = 2^{3-4m}\pi m C_{2m-2n-1}^{m-n} C_{2m+2n}^{m+n},\quad b_{mn}^{-} = \frac{2m-1}{2m} b_{mn}^{+}.$$

Sign $+$ and $v = r$, or $-$ and $v = \rho$, are adopted for $0 < \rho < r$ or $0 < r < \rho$, respectively.

Equation (13) can be solved approximately with the aid of the Gauss–Tchebyshev quadrature technique. We put

$$\varphi(\rho) = \sqrt{\frac{\rho}{a-\rho}}\,g(\rho),\quad \rho = 0.5a(1-\eta),\quad r = 0.5a(1-\xi).$$

Then, after discretization, eqn (13) is reduced to the set of linear algebraic equations

$$\sum_{j=1}^{N} A_{ij}g_j = f_i,\quad i = 1,\ldots,N$$

where

$$A_{ij} = H(\xi_i,\eta_j)\frac{2\pi}{2N+1},\quad g_j = g(\eta_j),\quad f_i = f(\xi_i),\quad \xi_i = \cos\frac{2i-1}{2N+1}\pi,$$

$$\eta_j = \cos\frac{2j}{2N+1}\pi,\quad i,j = 1,\ldots,N.$$

As a result we have solutions of eqn (10) for $j = 3, 5, 6, 70$ by formula (12) and for $j = 72$ as a set of values $g_j, j = 1,\ldots,N$.

Now the possibility exists of reconstructing the function $\varphi(s,\tau)$ in accordance with eqn (8). The stress intensity factor can be determined by means of a limiting procedure

$$K = \sqrt{\frac{2}{a}}\pi\lim_{r\to 1}\sqrt{1-r}\,\varphi(r,\theta) = \frac{2}{\pi}\sigma\sqrt{a}\,\kappa(\theta). \tag{14}$$

For the biperiodic crack system II

$$\kappa(\theta) = 1+4.969\varepsilon^3 - 1.364\varepsilon^5 + 24.68\varepsilon^6 - (37.53 - 40.88\cos 4\theta)\varepsilon^7. \tag{15}$$

Features should be noted of problems for the other surfaces enumerated in Section 2. For a periodic crack system (surface I) the sum with respect to k and m in eqn (5) is replaced by that with respect to k only. Here $R_k = R_0 - 2klq_1 + k^2l^2$, R_0, q_1 the same as in eqn (5). In expansion (7)

$$G_1 = \alpha_1[\rho(1.5\cos 2\tau + 0.5) - r(3\cos\theta\cos\tau) - \cos(\theta-\tau))], \quad \alpha_n = 2\sum_{n=1}^{\infty} k^{-(2n+1)},$$

and $G_j(\tau, \theta, \rho, r)$ includes $\cos 2n\tau$ for $n = 1, \ldots, j$.

For this reason, expressions

$$\varphi_{2j+3}(\rho, \tau) = \sum_{n=0}^{j-1} \varphi_{2j+3,n}(\rho)\cos 2n\tau, \quad j = 1, 2, \ldots,$$

should be adopted for $\varphi_{2j+3}(\rho, \tau)$. The expressions of the other functions φ_j are the same as before and further solution of the problem I is analogous to case II.

For break surface IV

$$R_n^2 = R_0^2 + (nh)^2, \quad G_1 = 2\pi\alpha_1(3\gamma-1)\rho, \quad G_2 = 2\pi\alpha_2(5\gamma-1)(\rho^3 + 2r^2\rho),$$
$$\gamma = 1/(1-2\nu), \quad \varphi_j = 0 \quad \text{for } j = 1, 2, 4$$

and

$$\varphi_j(\rho, \theta) = \rho P_j(\rho)/\sqrt{a^2-\rho^2} \quad \text{for } j = 3, 5, 6, \ldots,$$

where $P_j(\rho)$ is a polynomial of even degree not more than $j-3$. The solution of problem III (a laminated biperiodic break surface) combines the features of the solutions of problems I and IV, but the described method is applicable for a finite number N of layers. For $N \to \infty$ the series for the coefficients α_{nmj} in the expressions of the kernels G_{ij} are divergent.

4. RESULTS

Results of calculations of κ for ε and an angle θ between the chosen direction and the x-axis for a periodic (break surface I) and biperiodic (break surface II) crack system are given in Table 1 and in Fig. 2.

Figure 3 shows the dependence of the variable κ on parameters $\varepsilon = a/l$ and $\delta = a/h$ in a middle layer for break surface III involving three ($m = 3$) or seven ($m = 7$) layers. Table 2 gives the relationship between κ and the parameter δ for a periodic system of coplanar cracks (break surface IV).

Here κ is independent of θ and can be approximated by the polynomial

$$\kappa = 1 + 0.63\delta^3 - 42.51\delta^5 + 44\delta^6 + 203.5\delta^7. \quad (16)$$

The presented numerical results permit the following conclusions. For the periodic crack system I on a line through the centres, κ increases with ε (relative measure of crack). At the same time on a transverse line κ increases at first, but subsequently, for $\varepsilon \geqslant 0.3$, it decreases rapidly and becomes negative. This fact indicates closure of the break surface. The circular

Table 1. Stress intensity factor for periodic and biperiodic break surfaces I and II

	Surface I			Surface II	
ε	$\theta = 0$	$\pi/4$	$\pi/2$	0	$\pi/4$
0.1	1.002	1.002	1.002	1.005	1.005
0.2	1.011	1.012	1.013	1.041	1.040
0.3	1.044	1.037	1.020	1.150	1.132
0.4	1.179	1.040	0.830	1.412	1.277
0.5	1.718	0.874	−0.305	1.990	1.352
0.6	3.496	0.120	−4.455	3.213	0.924
0.7	8.396	−2.158	−16.239	5.655	−1.079

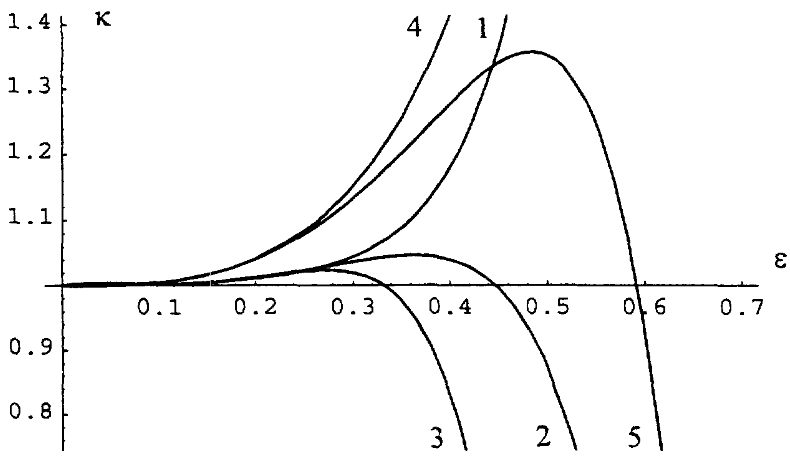

Fig. 2. Stress intensity factor. Periodic crack system: (1) $-\theta = 0$; (2) $-\theta = \pi/4$; (3) $-\theta = \pi/2$; biperiodic crack system: (4) $-\theta = 0$; (5) $-\theta = \pi/4$.

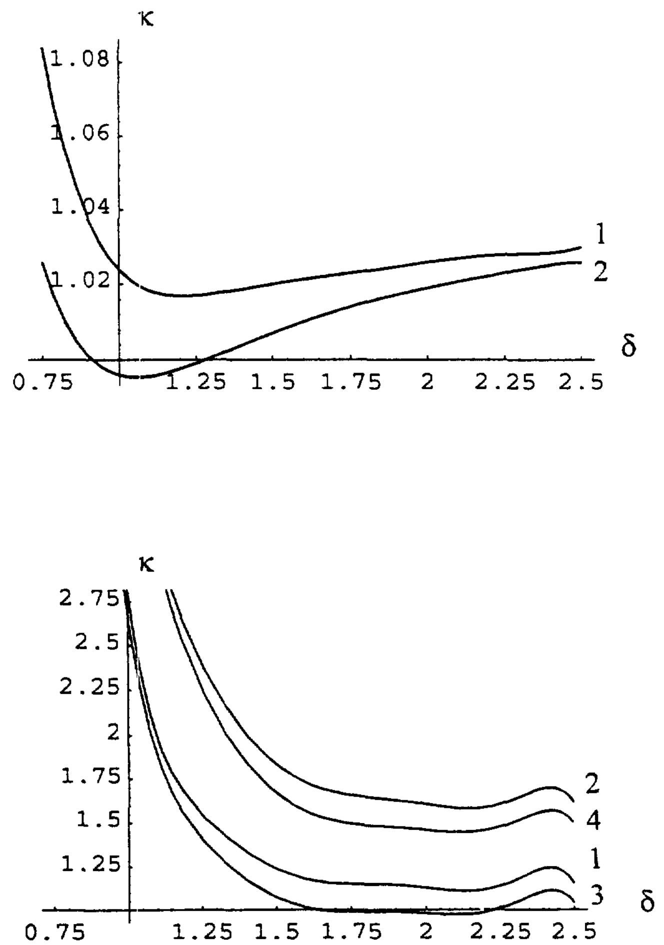

Fig. 3. Stress intensity factor for laminated crack system: (a) $\varepsilon = 0.2$, (1) $-m = 3$, $\theta = 0$, (2) $-m = 7$, $\theta = 0$; (b) $\varepsilon = 0.5$, (1) $-m = 3$, $\theta = 0$, (2) $-m = 3$, $\theta = \pi/4$, (3) $-m = 7$, $\theta = 0$, (4) $-m = 7$, $\theta = \pi/4$.

Table 2. Stress intensity factor for break surface IV

δ	0.1	0.2	0.3	0.4	0.5	0.6
κ	1.006	1.045	1.152	1.503	2.780	6.886

crack becomes a narrow elliptical crack stretched along the centre line. For the biperiodic break surface II the behavior of κ is similar to the previous case I, but increase of κ on the centre lines and decreasing on the bisectors is observed in this case to a lesser degree. For the laminated medium with break surface III, the dependence of κ on θ for small ε is also weak. In view of this the results for $\varepsilon = 0.2$ are shown in Table 2 for $\theta = 0$ only. For increasing ε this dependence is the same as in the earlier cases. Addition of layers leads to decrease of κ in the middle layer. Increase of the opening surface makes for reduction of the stresses. Finally, κ increases when the layers in break surfaces III and IV are brought closer together.

REFERENCES

Abramovitz, M. and Stegur, I. A. (1964) *Handbook of Mathematical Functions*. U.S. Government, Washington, D.C.

Aleksandrov, V. M. (1967) On the approximate solution of some integral equations of the theory of elasticity and mathematical physics. *Journal of Applied Mathematics and Mechanics* **31**(6), 1122–1136.

Banerjee, P. K. and Butterfield, R. (1981) *Boundary Element Methods in Engineering Science*. McGraw-Hill, London.

Kurshin, L. M. and Suzdalnitsky, J. D. (1975) Stress state of an elastic plane weakened by an infinite series of longitudinal–transverse cracks. *Journal of Applied Mechanical and Technical Physics* **6**(5), 826–832.

Kurshin, L. M. and Suzdalnitsky, J. D. (1978) Stress state in a plane with curvilinear cut. *Mechanics of Solids* **13**(1), 170–173.

Liebowitz, H. (1968) *Fracture. An Advanced Treatise*, Vol. 2. Academic Press, New York.

Qin, T. Y., Yoe, J. C. and Tang, R. J. (1995) Application of the hypersingular integral equations method to the interaction between two parallel planar cracks in 3D elasticity. *Acta Mechanica Sinica* **8**(1), 64–74.